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On Kaplun's optimal co-ordinates

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Optimal co-ordinates were introduced by Kaplun (1954) as a result of his study on the role of co-ordinate systems in boundary-layer theory. In this paper the basic ideas of optimal co-ordinates are examined and many restrictions of Kaplun's optimal co-ordinates are removed. His rule for constructing optimal co-ordinates is found to apply unaltered to axisymmetric flows, to flows with oncoming streams containing vorticity, to free boundary layers such as jets, to free-convection flows and to compressible flows. It is also extended to unsteady boundary layers, and to three-dimensional boundary layers using a pair of stream functions.

1. Introduction

One of the first questions that an investigator asks upon having formulated a physical problem is: what co-ordinate system should be employed to solve the problem? The choice of a particular system is often guided by the geometry, at other times by convenience, and sometimes by the physics of the problem. The method of optimal co-ordinates, introduced by Kaplun (1954) in his novel papers on boundary-layer theory and singular-perturbation techniques, is a means of including the boundary-layer physics in the choice of the co-ordinate system.

The purpose of this paper is two-fold. Firstly, it seeks to emphasize the generality of Kaplun's optimal-co-ordinates theorem. Secondly, it seeks to review the efforts that have been made to utilize and describe optimal co-ordinates. Kaplun's (1954) paper was limited to the study of laminar, incompressible, steady, two-dimensional unseparated flow past a solid body, with an irrotational outer flow. No attempts were made to investigate the generality of Kaplun's results until the efforts of Legner (1971). The recognition of Kaplun's work may once again be enhanced by the realization that his rule for constructing optimal co-ordinates applies unaltered to axisymmetric flows, flows with oncoming streams containing vorticity, free boundary layers (such as jets), free-convection flows, compressible flows, and to uncoupled and coupled thermal boundary layers. The reasons behind these generalizations are described in this paper. In addition, rules for finding optimal co-ordinates are extended to unsteady boundary layers and to three-dimensional boundary layers using Clebsch's pair of stream functions.

Section 2 puts together for the first time an extensive examination of the literature on the use of Kaplun's optimal co-ordinates. Some of the less important references are omitted here (cf. Legner (1971) for a more complete review); we summarize the critical efforts. We also point out some of the misconceptions that have crept into the literature. Furthermore, some connections between optimal co-ordinates and the determination of uniformly valid asymptotic solutions are described.

The review in § 2 is followed in § 3 by a discussion of the generality of Kaplun's work. Sections 4 and 5 describe the extensions to unsteady and three-dimensional boundary layers, respectively. Optimal co-ordinates are determined utilizing the analysis developed herein for two specific examples in § 6. One example is the rotational flow past a flat plate (Murray 1961) and the other is the initial boundary-layer development on a circular cylinder started from rest (Wang 1967). Concluding remarks are presented in § 7.

2. Review of optimal co-ordinates

We first summarize Kaplun (1954). Utilizing the concepts of distinct limit processes in separate (but overlapping) regions of a viscous boundary-layer flow problem Kaplun argued that since the boundary-layer limit, which depends upon the co-ordinate-system used, leads to the successive boundary-layer system of equations it must also be true that the boundary-layer solutions are co-ordinate-system-dependent. This realization, as well as the co-ordinate-system independence of the sequential potential flow problems, enables him to look for a co-ordinate system such that the classical (first-order) boundary-layer problem contains both the basic inviscid flow and the second inviscid flow term – the displacement flow. He finds such an optimal co-ordinate system (his theorem 2). Kaplun also delineates the relationship between boundary-layer solutions in two different co-ordinate systems (his theorem 1). Three examples from classical boundary-layer theory illustrated the content of the two theorems.

The most authoritative presentation of Kaplun's work is given by Lagerstrom (1964) in his survey of laminar-flow theory. Van Dyke (1964) gives a good discussion of the effect of changing the co-ordinate system on the classical Blasius problem; in particular, he compares solutions for optimal, semi-optimal, and ordinary† co-ordinate systems. Howarth (1959) details the main ideas and remarks that the importance of Kaplun's work resides upon improving the boundary-layer approximation. These are general references.

Most of the applications of Kaplun's ideas have been related to the choice of a specific co-ordinate system. Parabolic co-ordinates have been most popular. They are the optimal co-ordinates for the classical Blaius flat-plate problems as demonstrated by Kaplun. Murray (1961, 1965*a*, *b*) uses parabolic co-ordinates in his various studies of the flat-plate problem. Davis (1967) employs parabolic co-ordinates for his series-truncation analysis near the flat-plate leading edge. Some time later, van deVooren and Dijkstra (1970) examined the same leading-edge problem more carefully by numerically integrating the Navier–Stokes equations written in parabolic co-ordinates by means of a finite-difference technique. They argued that the optimal co-ordinates obtained from boundary-layer theory should also be preferable for the complete Navier–Stokes equations. This idea is specifically corroborated by some exact solutions of the Navier–Stokes equations in which the boundary-layer approximation optimal co-ordinates lead to the co-ordinates of the exact solution (e.g. Hiemenz's (1911) problem and Cartesian co-ordinates). Van Dyke (1970) uses parabolic co-ordinates in his article on channel entry in which he conjectures that parabolic

† The use of an ordinary co-ordinate system provides a boundary-layer solution which, in general, only matches with the outer flow.

co-ordinates are presumably nearly optimal for a cascade of flat plates. In two related papers, Wilkinson (1960) and Townsend (1965) solve the steady three-dimensional boundary-layer flow past a flat plate with a parabolic leading edge for the incompressible and compressible flow situations, respectively. They discover that a degenerate form of paraboloidal co-ordinates provide boundary-layer velocity components that are optimal. These results reflect the intimate relationship between 'generalized' parabolic co-ordinates and the first-order laminar boundary layer on flat plates.

Some investigators have either been naive about the content of Kaplun's theorems or simply misleading. As a first example, we consider Kadambi's (1969) examination of the natural convection of a heated plate in a gravity field. Without discussion, Kaplun's theorems are utilized to obtain non-parabolic optimal co-ordinates; no mention is made of the fact that the boundary-layer variable for problems driven by the boundary layer (with the first-order potential stream function equal to a constant) is arbitrary, or of the fact that there is no *a priori* reason to assume that the coupled temperature expansion plays no direct part in the determination of the optimal co-ordinates. Schultz-Grunow & Henseler (1968) confuse optimal and semi-optimal co-ordinates. A more serious misinterpretation is exemplified by Mills (1965). He confuses optimal co-ordinates and the von Mises transformation in his paper on flow in a square cavity. Lagerstrom (1964, p. 214) has anticipated this difficulty and remarks in a footnote that the von Mises transformation of the derived boundary-layer equations may simplify the equations (and solutions) but it does not alter the flow field obtained from the equations.

The ideas of optimal co-ordinates have also become enmeshed with other singular perturbation techniques. The method of Lighthill, as well as the use of parabolic co-ordinates (optimal for the semi-infinite flat plate), is considered by Goldburg & Cheng (1961) in their study of the trailing-edge boundary layer. They discover an anomaly in the application of these two methods; the methods differ by an order of magnitude in the estimation of the extent of the upstream influence for the trailing edge. They speculate that neither approach is correct. Recent examinations of the trailing-edge problem (using triple decks) have shown that the full Navier–Stokes equations must be used near the edge; hence, the Lighthill method which utilizes the boundary-layer equations cannot be valid and the use of parabolic co-ordinates (Kaplun-optimal) will not be completely effective either, since the trailing edge will necessitate modifications to the optimal co-ordinates. That is, second-order boundary-layer theory, which presumably describes the Navier–Stokes situation more accurately, will alter Kaplun's optimal co-ordinates. Segel (1960) related the ideas of uniform asymptotic expansions to optimal co-ordinates. A non-fluid-mechanical application of Kaplun's ideas was considered by Zauderer (1970), who introduced two co-ordinate systems in his study of diffraction problems. Singular perturbations and optimal co-ordinates are closely related. O'Malley (1968) in his survey of singular perturbations mentions Kaplun's optimal co-ordinates for Oseen flow past a flat plate (parabolic co-ordinates) but illustrates an alternative way of finding the exact solution by employing a 'boundary-layer stretching co-ordinate' which increases the number of independent variables by one.

This review has been, of necessity, brief. It has sought to emphasize some of the more significant applications and discussions of optimal co-ordinates. It must be evident to the reader that, from the above survey of the literature two important points can be made: firstly, no-one has heretofore used optimal co-ordinates to solve a problem which could not be solved in any other manner, i.e. there has been no 'critical' application; secondly, no one has critically examined the generality of Kaplun's paper and whether it is possible to extend the idea to higher approximations. This paper emphasizes the generality of Kaplun's (1954) work.

3. The generality of Kaplun's theorem 2

The discussion in this paper will be limited to the realm of boundary-layer theory that includes the flow due to the displacement thickness. An important aspect of the boundary-layer approximation is its coordinate-system dependence. The concept of co-ordinate-system dependence is explained below by appealing to a variety of approximations to the flow equations. The governing fluid-dynamical equations, the Navier–Stokes equations, describe the behaviour of the physics of fluids and, hence, are necessarily co-ordinate-system independent. These equations can be written in vector form and thereby have vector (or co-ordinate-system-independent) solutions. Approximations to the Navier–Stokes equations may or may not have vector solutions. This depends upon the character of the approximation. The incompressible inviscid approximation (the Euler equations) is a vector approximation since the entire viscous term is neglected, leaving a system of equations that is vectorial. The first approximation in low-Reynolds-number flow (the Stokes equations) is also a vector approximation because in this case the entire convective term is neglected. These vector approximations must have vector solutions. On the other hand, the boundary-layer approximation, wherein viscous diffusion is preferentially neglected in the stream direction as opposed to the boundary-layer direction, is a non-vector approximation. Boundary-layer solutions are therefore non-vector (or co-ordinate-system-dependent) solutions. The genesis of optimal co-ordinates resides in the last statement.

A second important point regarding optimal co-ordinates relates to the generality of the method. The specific details of the boundary-layer equations under various approximations relating to the dimensionality, the time-dependence or the compressibility, for example, are *not* relevant to the matter of optimal co-ordinates. It is only the structure of boundary-layer theory as derived rationally using the method of matched asymptotic expansions (cf. Van Dyke 1962) which is significant. More specifically, it will be shown that the exact satisfaction of the (unapproximated) continuity equation using the stream function(s) is a critical element in the determination of optimal co-ordinates.

The classical fluid-mechanical system of equations can be written in generalized tensor form following Lagerstrom (1964). In this form one can consider a general curvilinear co-ordinate system ξ^i. Legner (1971) found it convenient to put this system into dimensionless form; the important parameter of this system is the Reynolds number $R \equiv \epsilon^{-2} \equiv \rho_r U_r L_r/\mu_r$ where all quantities with subscript r are effective reference quantities. As remarked previously, there is no need to write out the dynamical equations; however, it is essential to utilize the continuity equation. This equation (in Lagerstrom form) is

$$\frac{1}{\delta}\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u^i)}{\partial \xi^i} = 0, \tag{1}$$

where ρ is a *density* in the sense of tensor analysis, and u^i are the contravariant components of the velocity vector and $\delta = t_r U_r / L_r$ is a constant. One can write the preceding equations in the form Kaplan used by setting $\rho = g^{\frac{1}{2}}\tilde{\rho}$ and $\tilde{u}^i = u^i g^{\frac{1}{2}}$ where $\tilde{\rho}$ is now an absolute scalar, $\tilde{u}^i$ is a contravariant vector density, and $g^{\frac{1}{2}}$ is the square root of the absolute value of g_{ij}, the metric tensor. In order to distinguish the possible situations when the continuity equation can be satisfied identically, we must consider three separate aspects: time dependence (t), compressibility (ρ), and dimensionality (ξ^i, u^i). The understanding of these aspects leads to only three types of situation (one-space-dimension problems are not relevant to the discussion).

The possibilities are outlined below.

Type 1. *One stream function* (2-*dimensional*):
(a) steady, incompressible; (b) unsteady, incompressible; (c) steady, compressible.

Type 2. *Two stream functions* (3-*dimensional*):
(a) steady, incompressible; (b) unsteady, incompressible; (c) steady, compressible.

Type 3. *Restricted one* (*two*) *stream functions*:
(a) unsteady, compressible (2-dimensional); (b) unsteady, compressible (3-dimensional).

Little comment is necessary for the problems of type 1. Problems analyzed using one stream function are familiar to most investigators. The problems encompassing type 2 are more complicated. Even though researchers have analyzed three-dimensional problems (e.g. using velocity components and pressure) there appear to be few analyses in the literature using a pair of stream functions. The reason for this is the complexity (and nonlinearity) introduced in defining the two-stream functions; that is, whereas a velocity component is given by one derivative of the stream function in type 1, it is given by a combination of four derivative terms in type 2. Two stream functions for three-dimensional flows are described in § 5. The problems of type 3 are even more complicated. We refer to these problems as *restricted* because even though the continuity equation can be satisfied using generalizations of the Howarth–Dorodnitsyn transformation (see Stewartson 1964, p. 123 for the two-dimensional case), we must recognize that these transformations belong to first-order boundary-layer theory only.

These preliminary remarks may appear unrelated to the matter of optimal co-ordinates, but it will become evident that the rational development of boundary-layer theory must utilize the stream function. In a real sense, the stream surfaces of the oncoming flow and of the displacement flow are natural physical entities. How do these remarks relate to Kaplun (1954)? We make the connections to Kaplun by quoting a number of his results using our notation. If we develop the theory using the stream function, we are concerned with two terms of the Euler expansion

$$\psi \sim \psi_1 + \epsilon\psi_2 + \dots, \tag{2}$$

and a single term from the boundary-layer expansion

$$\psi \sim \epsilon\Psi_1 + \dots . \tag{3}$$

Further, if $\eta = 0$ defines a solid wall (or some line or axis for free flows) and ξ is a streamwise co-ordinate, then it can be shown using the *limit-matching principle* of Van Dyke (1964, p. 90) that the boundary-layer expansion behaves as follows near the edge of the layer:

$$\psi \sim \epsilon\Psi_1 \sim \psi_{1\eta_w}\eta + \epsilon\psi_{2w} + \dots, \tag{4}$$

as $\bar{\eta} \equiv \eta/\epsilon \to \infty$. We retain only two terms, anticipating matching with (2). Optimal co-ordinates are defined by a much stronger condition than matching. They require the boundary-layer-derived flow to *contain* the Euler flow to some order. Specifically, in order for the first-order boundary-layer solution to contain the external flow (i.e. the main flow ψ_1 plus the flow due to the displacement thickness ψ_2) we must satisfy the following relations:

$$\psi_1 \equiv \psi_{1\eta w}\eta, \quad \psi_2 \equiv \psi_{2w}. \tag{5a, b}$$

Note that the subscript w refers to the limit of an outer flow variable as one tends to the wall ($\eta \to 0$). Hence, $\psi_{2w} = \psi_{2w}(\xi, 0)$. The co-ordinates obtained from the inversion of (5) using Kaplun's correlation theorem provide optimal co-ordinates. This theorem (no. 1) states that if the relationship between two co-ordinate systems is $\xi = \xi(\gamma, \sigma)$ and $\eta = \eta(\gamma, \sigma)$, with $\eta = 0$ and $\sigma = 0$ prescribing the body surface, then the equivalent boundary-layer solution in the (γ, σ)-system is found by replacing ξ by $\xi(\gamma, 0)$ and η by $(\partial\eta/\partial\sigma)_{\sigma=0}\sigma$. These expressions can be written in the following form:

$$\xi = g_1(\gamma), \quad \eta = g_2(\gamma)\sigma, \tag{6a, b}$$

where g_1 and g_2 are simply functions of γ. These functions can be chosen to suit the investigator's needs; if we substitute (6) into (5), we find

$$\psi_1 = \psi_{1\eta w}[g_1(\gamma)]\,g_2(\gamma)\,\sigma, \quad \psi_2 = \psi_{2w}[g_1(\gamma)]. \tag{7a, b}$$

Using our stated freedom, we select g_1 such that $\gamma = \psi_2$ and $g_2(\gamma) = 1/\psi_{1\eta w}[g_1(\gamma)]$ so that $\sigma = \psi_1$. This particular set of γ and σ is a pair of optimal co-ordinates. The most general set is given by utilizing (6); hence one obtains Kaplun's important result (theorem no. 2)

$$\xi_{\text{opt}} = f_1(\psi_2), \quad \eta_{\text{opt}} = \psi_1 f_2(\psi_2). \tag{8a, b}$$

With these results as background, we can proceed to discuss the generality of Kaplun's results.

(i) *Axisymmetric, incompressible, steady flow*

Consider the continuity equation (1). Upon introducing a scalar density $\tilde{\rho}$ using $\rho = g^{\frac{1}{2}}\tilde{\rho}$, and considering $\tilde{u}^i = g^{\frac{1}{2}}u^i$ as contravariant-vector-density components, we can write (1) as

$$\frac{\partial \tilde{u}^i}{\partial \xi^i} = 0. \tag{9}$$

Equation (9) is preferable to forms containing $g^{\frac{1}{2}}$. If we now assume $\partial/\partial\xi^3$ is identically zero, (9) becomes

$$\frac{\partial \tilde{u}^1}{\partial \xi^1} + \frac{\partial \tilde{u}^2}{\partial \xi^2} = 0. \tag{10}$$

Equation (10) is valid for both two-dimensional flow and axisymmetric flow; thus we satisfy (10) identically with a *two-co-ordinate stream function* ψ, i.e.

$$\tilde{u}^1 = \frac{\partial \psi}{\partial \xi^2}, \quad \tilde{u}^2 = -\frac{\partial \psi}{\partial \xi^1}. \tag{11}$$

Since the definition of ψ in (11) is independent of the assumption of axisymmetry, the axisymmetry (with $g_{33} \neq 1$) manifesting itself only in the boundary-layer momentum and energy equations, and since the optimal-co-ordinate development (as outlined above) is independent of these equations, it follows that Kaplun's result applies unaltered to axisymmetric flows.

(ii) *Euler flow with vorticity*

It is not necessary to restrict the outer flow to be governed by Laplace's equation (irrotational flow). The rotational vorticity equation has solutions which are independent of the co-ordinate system. These vector problems, which govern the outer flow, have vector solutions and they can be matched in the usual asymptotic sense to the boundary-layer solution. Consequently, Kaplun's result applies to flows with rotational external streams.

(iii) *Free jets*

Free jets form a class of boundary-layer flows where the outer flow is zero; i.e. $\psi_1 \equiv 0$. Jets differ from surface boundary layers in that a symmetry condition and a global integral condition replace the typical surface conditions. These two conditions *do not* alter the optimal coordinates development. We must, however, examine (7). Equation (7*a*) is satisfied identically, independently of co-ordinate-system concerns. It then follows that the optimal co-ordinates are given by

$$\xi_{\text{opt}} = f_1(\psi_2), \tag{12a}$$

$$\left.\begin{array}{c} \eta_{\text{opt}} \text{ is arbitrary} \\ \text{(with the proviso that } (\xi_{\text{opt}}, \eta_{\text{opt}}) \text{ define a co-ordinate system).} \end{array}\right\} \tag{12b}$$

(iv) *Free convection*

Natural or free convection boundary layers are another class of flows for which $\psi_1 \equiv 0$; however, natural convection phenomena are considerably more complicated than incompressible free jets since the momentum equation is coupled to the energy equation through the Boussinesq buoyancy term. This complexity does *not* manifest itself in the determination of optimal co-ordinates because, as demonstrated in § 3 (v), the addition of a dependent variable does not alter the stream-function-determined optimal co-ordinates. It then follows that the optimal co-ordinates for natural convection are given by (12*a*, *b*). Kadambi (1969) was apparently the first to use $\xi = f_1(\psi_2)$ without explanation in order to determine optimal co-ordinates for a specific free-convection problem. We note that free-jet flows and natural convection flows are characteristic of situations wherein the boundary layer drives the flow. Therefore, it is not surprising that the optimal coordinates are determined using the same equations.

(v) *Coupled dependent variables*

Consider a flow problem with dependent variables such as the temperature, density, pressure, concentration, etc., coupled to the stream function ψ. We will show that stream-function considerations dominate the optimal co-ordinates. Let us assume that we have some hypothetical flow problem which can be reduced to the solution of two simultaneous equations in ψ and ϕ, where ψ is the stream-function and ϕ represents any other dependent variable. Suppose that the exact equations are $a(\psi, \phi; \epsilon) = 0$, $b(\psi, \phi; \epsilon) = 0$ and assume that all boundary conditions are satisfied implicitly in the subsequent discussion. The inner limit applied to the problem provides $A_1(\Psi_1, \Phi_1) = 0$ and $B_1(\Psi_1, \Phi_1) = 0$; the outer limit provides $a_1(\psi_1, \phi_1) = 0$,

$b_1(\psi_1, \phi_1) = 0$ in the first approximation and $a_2(\psi_2, \phi_2) = 0$, $b_2(\psi_2, \phi_2) = 0$ in the second approximation. Let the boundary-layer solution be given by $\Psi_1(\rho, \bar{\sigma})$ and $\Phi_1(\rho, \bar{\sigma})$. Using our *stream-function-determined* optimal co-ordinates $(\xi^*, \bar{\eta}^*)$, we can rewrite the solution as $\Psi_1^*(\xi^*, \bar{\eta}^*)$ and $\Phi_1^*(\xi^*, \bar{\eta}^*)$. By virtue of its construction (this is our hypothesis), Ψ_1^* now contains $\psi_1 + \epsilon\psi_2$ instead of merely matching with $\psi_1 + \epsilon\psi_2$, as Ψ_1 does. Consider Φ_1^*. Suppose Φ_1^* does not contain $\phi_1 + \epsilon\phi_2$ as $\bar{\eta}^*$ becomes large, but instead becomes $\hat{\phi}_1 + \epsilon\hat{\phi}_2$. The quantity $\hat{\phi}_1 + \epsilon\hat{\phi}_2$ implies from $(a_1 = 0, b_1 = 0)$ that the external stream function is $\hat{\psi}_1 + \epsilon\hat{\psi}_2$. This, however, contradicts our hypothesis that Ψ_1^* contains $\psi_1 + \epsilon\psi_2$. Hence, we conclude that Φ_1^* contains $\phi_1 + \epsilon\phi_2$. The same statement applies to other additional dependent variables. The above arguments are summarized as follows. If the boundary-layer stream function, as a dependent variable, contains the outer field in the optimal-co-ordinate sense, then any coupled boundary-layer dependent variable must also contain the outer field in an optimal sense due to the connection among the dependent variables.

(vi) *Compressible flows*

We now examine a real compressible flow with significant density variations. Consider the steady compressible continuity equation $\partial(\tilde{\rho}\tilde{u}^i)/\partial\xi^i = 0$. For the two-co-ordinate system, this equation reads $\partial(\tilde{\rho}\tilde{u}^1)/\partial\xi^1 + \partial(\tilde{\rho}\tilde{u}^2)/\partial\xi^2 = 0$. If we let $\tilde{\rho}\tilde{u}^1 = n^1 \equiv m$ and $\tilde{\rho}\tilde{u}^2 = n^2 \equiv n$, we then obtain

$$\frac{\partial m}{\partial \xi^1} + \frac{\partial n}{\partial \xi^2} = 0. \tag{13}$$

The introduction of m and n allows us to introduce a stream function ψ as follows:

$$m = \frac{\partial \psi}{\partial \xi^2}, \quad n = -\frac{\partial \psi}{\partial \xi^1}. \tag{14}$$

Clearly, (14) satisfy (13) exactly; the remaining compressible equations can then be written in terms of ψ, ρ, and T. Since the differential equations were not employed for the incompressible problem, and since the stream-function expansion dominates the optimal co-ordinates as argued in the preceding discussion, we assert that the structure of the compressible-stream-function expansions dominates the compressible-flow optimal co-ordinates.

Proceeding as before for the incompressible case we write $\psi \sim \psi_1 + \epsilon\psi_2 + \ldots$, in the outer field, and $\psi \sim \epsilon\Psi_1 + \ldots$, in the inner field. We can write these expansions since the compressible boundary layer is known to be qualitatively similar to the incompressible one. Note that we can freely manipulate the density $\tilde{\rho}$ since it can never be zero. The basic analysis of the incompressible development applies without any changes. Hence, $(8a, b)$ *provide the optimal co-ordinates for the compressible boundary-layer problem.*

The extension of $(8a, b)$ to compressible flow is a remarkable result. The other dependent variables of the problem are intimately connected to the stream function; further, the result is independent of the physics: gas law, viscosity–temperature relation, heat–conductivity–temperature relation, etc. That is, the result is general and free of any difficulties if the flow does not contain shock waves.

4. Unsteady boundary layers

Unsteady boundary-layer theory introduces a third independent variable into the development. This theory has not been analysed to the extent of two-co-ordinate flow. There are two types of problem in the literature – the *small-time formation* problem (or starting problem) and the *periodic flow* problem. The difference between such problems is the relative importance of the nonlinear convective terms. Starting-process problems are linear, whereas developed unsteady flows are nonlinear. For the purposes of determining optimal co-ordinates, we need not distinguish between the two types of problem. We proceed directly to the heart of the analysis, i.e. to invert (5*a*, *b*). We recognize that for incompressible unsteady flows we can satisfy the continuity equation exactly with expressions (11). In the present context, it is understood that $\psi = \psi(\xi, \eta, \tau)$ where τ is the time co-ordinate. Furthermore, the terms with subscript w (evaluated at the wall) must be regarded as functions of both ξ and τ. To be specific, we write out (5*a*, *b*) as

$$\psi_1 = \psi_{1\eta_w}(\xi, \tau)\,\eta, \quad \psi_2 = \psi_{2w}(\xi, \tau). \tag{15a, b}$$

If we now introduce the *correlation co-ordinate transformation* expanded for three co-ordinates, letting ω be the new time co-ordinate, then the generalization of (6*a*) and (6*b*) is

$$\xi = g_1(\gamma, \omega), \quad \eta = g_2(\gamma, \omega)\,\sigma, \quad \tau = g_3(\gamma, \omega). \tag{16a, b, c}$$

The introduction of (16) into (15*b*) permits us to select

$$g_2(\gamma, \omega) = 1/\psi_{1\eta_w}[g_1(\gamma, \omega), g_3(\gamma, \omega)]$$

so that $\sigma = \psi_1$. We saw the same result previously. In order to determine the optimal co-ordinates, we must satisfy (15*b*). This cannot be done in general since the substitution of (16) into (15) represents a system of two equations in three unknowns. This dilemma is resolved by assuming that either the time dependence or the surface-co-ordinate dependence is passive.

(i) *Time-dependence passive* ($g_3(\gamma, \omega) = \omega$). Then the optimal co-ordinates become

$$\xi_{\text{opt}} = f_1(\psi_2, \omega), \quad \eta_{\text{opt}} = \psi_1 f_2(\psi_2, \omega), \quad \tau_{\text{opt}} = \omega. \tag{17a, b, c}$$

(ii) *Surface-dependence passive* ($g_1(\gamma, \omega) = \gamma$). Here the optimal co-ordinates become

$$\xi_{\text{opt}} = \gamma, \quad \eta_{\text{opt}} = \psi_1 f_2(\gamma, \psi_2), \quad \tau_{\text{opt}} = f_3(\gamma, \psi_2), \tag{18a, b, c}$$

where f_1, f_2, f_3 are arbitrary functions. Note that Legner (1971) was able to obtain more general optimal co-ordinates for higher-order unsteady boundary-layer approximations. These results, (17) and (18), should not appear unexpected. The time co-ordinate appears to play the same role as the surface co-ordinate in optimal co-ordinates. This parallelism between the time evolution of, say, the Rayleigh-problem boundary layer and the spatial development of the Blasius boundary layer is well known. It is naturally evident in the present context.

As remarked in § 3, a stream function can be defined for unsteady two-dimensional compressible flow (see Stewartson 1964, p. 123). This circumstance allows the determination of optimal co-ordinates as described above with the density playing an inessential part. It must, however, be emphasized that the generalization of the Howarth–Dorodnitsyn transformation as described by Stewartson will *only* permit the determination of optimal co-ordinates for first-order boundary-layer theory.

5. Three-dimensional boundary layers

The elements of three-dimensional boundary-layer theory have been developed by Legner (1971) using a pair of stream functions, ψ and χ. For incompressible flow, the velocity vector satisfying the continuity equation $\operatorname{div}\mathbf{q} = 0$ identically is

$$\mathbf{q} = \operatorname{grad}\psi \times \operatorname{grad}\chi. \tag{19}$$

The co-ordinates (ξ, η, ζ) will be used, with $\eta = 0$ again denoting the surface of interest and ζ the cross-stream or third co-ordinate. The boundary conditions at infinity are $\psi = \psi_{\mathrm{E}}(\xi, \eta, \zeta)$ and $\chi_{\mathrm{E}}(\xi, \eta, \zeta)$, where the subscript E denotes a potential (or Euler) flow. Before we write the surface conditions, let us write out (19) for the three components of velocity when (19) is written in tensor form, i.e. $\tilde{u}^i = e^{ikl}\psi_{,k}\chi_{,l}$:

$$\tilde{u}^1 = \frac{\partial\psi}{\partial\eta}\frac{\partial\chi}{\partial\zeta} - \frac{\partial\psi}{\partial\zeta}\frac{\partial\chi}{\partial\eta}, \tag{20a}$$

$$\tilde{u}^2 = -\frac{\partial\psi}{\partial\xi}\frac{\partial\chi}{\partial\zeta} + \frac{\partial\psi}{\partial\zeta}\frac{\partial\chi}{\partial\xi}, \tag{20b}$$

$$\tilde{u}^3 = \frac{\partial\psi}{\partial\xi}\frac{\partial\chi}{\partial\eta} - \frac{\partial\psi}{\partial\eta}\frac{\partial\chi}{\partial\xi}. \tag{20c}$$

These relations are considerably more complex than the single-stream-function relations. At the (impermeable solid) surface we desire $\psi = 0$ at $\eta = 0$. This condition implies that the surface of interest is a stream surface; it also immediately gives

$$\frac{\partial\psi}{\partial\xi} = \frac{\partial\psi}{\partial\zeta} = 0 \quad \text{at} \quad \eta = 0. \tag{21}$$

The velocity components thus read $\tilde{u}^1 = (\partial\psi/\partial\eta)(\partial\chi/\partial\zeta), \tilde{u}^2 = 0, \tilde{u}^3 = -(\partial\psi/\partial\eta)(\partial\chi/\partial\xi)$ All of these relations indicate that if the $\eta = 0$ surface is a stream surface, there is a flow *on* the surface ($\tilde{u}^1, \tilde{u}^3 \neq 0$) and no flow *through* the surface ($\tilde{u}_2 \equiv 0$). The no-slip condition requires $\tilde{u}^1 \equiv \tilde{u}^3 \equiv 0$ at the surface. According to the previous relations for $\tilde{u}^1$ and $\tilde{u}^3$, this could be accomplished in two ways: we could require $\partial\psi/\partial\eta = 0$ or $\partial\chi/\partial\zeta = \partial\chi/\partial\xi = 0$ at $\eta = 0$. The latter conditions imply that $\chi =$ constant ($=$ zero). If $\chi = 0$ on the surface, $\tilde{u}^1 = \tilde{u}^2 = \tilde{u}^3 = 0$; however, this would appear to represent a degenerate situation where the notion of streamlines is lost. Furthermore, if we were to specialize our problem to two dimensions, where $\chi = \zeta$, we have $\partial\chi/\partial\zeta \neq 0$. This would violate the above $\partial\chi/\partial\zeta = 0$ condition. In order to avoid this situation, we choose the first of the two conditions, i.e. $\partial\psi/\partial\eta = 0$ at $\eta = 0$. This will make all velocity components zero at the surface. Summarizing then, the surface conditions are

$$\psi = \frac{\partial\psi}{\partial\eta} = 0, \quad \chi \neq \text{const.} \quad \text{at} \quad \eta = 0; \chi, \chi_\xi, \chi_\eta, \chi_\zeta \quad \text{bounded.} \tag{22}$$

We could be more specific about the conditions on χ; however, this is not necessary since we develop the optimal co-ordinates without recourse to other dependent variables, as remarked previously. That is, we can work with ψ alone.

The outer expansion for ψ will be

$$\psi \sim \psi_1 + \epsilon\psi_2 + \ldots, \tag{23}$$

and the inner expansion will be

$$\psi \sim \epsilon \Psi_1(\xi, \eta, \zeta) + \dots . \tag{24}$$

The extra third co-ordinate ζ will play the same role as the time co-ordinate in unsteady boundary-layer theory. Consequently, it becomes easy to invert the conditions for optimal co-ordinates; they read

$$\psi_{1\eta_w}(\xi, \zeta)\,\eta = \psi_1, \quad \psi_{2w}(\xi, \zeta) = \psi_2. \tag{25a, b}$$

Equations (25a, b) are completely analogous to the relations derived for (15a b). We can also invert these relations, as was done for unsteady boundary-layers, in order to obtain optimal co-ordinates. Two cases arise again because both ξ and ζ are 'timelike'.

(i) *Cross-stream dependence ζ passive.* The optimal co-ordinates become

$$\xi_{\text{opt}} = f_1(\psi_2, \lambda), \quad \eta_{\text{opt}} = \psi_1 f_2(\psi_2, \lambda), \quad \zeta_{\text{opt}} = \lambda. \tag{26a, b, c}$$

(ii) *Streamwise dependence ξ passive.* The optimal co-ordinates become

$$\xi_{\text{opt}} = \gamma, \quad \eta_{\text{opt}} = \psi_1 f_2(\psi_2, \gamma), \quad \zeta_{\text{opt}} = f_3(\psi_2, \gamma), \tag{27a, b, c}$$

where once again f_1, f_2, and f_3 are arbitrary functions. Note that the transformation of co-ordinates utilized in this three-dimensional situation was $\xi = \xi(\gamma, \sigma, \lambda)$, $\eta = \eta(\gamma, \sigma, \lambda)$, and $\zeta = \zeta(\gamma, \sigma, \lambda)$. A more general set of optimal co-ordinates for three-dimensional boundary layers is obtained (Legner 1971) when higher-order boundary-layer approximations are considered.

6. Specific examples

In this section optimal co-ordinates are determined for two specific examples utilizing the analysis developed in the present paper. The examples illustrate the manner in which optimal co-ordinates are selected for two generalizations of Kaplun's work: generalization to a rotational outer flow and generalization to unsteady boundary layers.

Rotational flow past a flat plate

Murray's (1961) solution for the boundary-layer on a flat plate in a stream with uniform shear is investigated. The problem has two features that are important to optimal co-ordinates. First, the Euler flow field is rotational; second, Kaplun's optimal co-ordinates for the flat plate without shear (parabolic co-ordinates) were used to develop the solution. These (parabolic) co-ordinates are *not* optimal when the outer flow is rotational. The free-stream velocity in the plate direction is

$$u = U + \Omega_0 y, \tag{28}$$

where all quantities are dimensional. If we scale velocities by U, lengths by an artificial length l, stream function by Ul, and define a vorticity number $N \equiv \Omega_0 l/U$ and Reynolds number $R \equiv Ul/\nu$, then (28) in dimensionless form reads

$$\tilde{u} = 1 + N\tilde{y}. \tag{29}$$

The elements of boundary-layer theory in terms of dimensionless parabolic co-ordinates defined by $\tilde{x} + i\tilde{y} \equiv (\gamma + i\delta)^2$ are as follows:

$$\psi_1 = 2\gamma\delta + 2N\gamma^2\delta^2, \quad \psi_2 = -\beta\gamma, \tag{30a, b}$$

where β = Blasius-function constant. The boundary-layer expansion with $\epsilon = R^{-\frac{1}{2}}$ is, to first order,

$$\psi \sim \epsilon[\gamma f(\bar{\delta})+\ldots], \tag{31}$$

where $f'''+ff''=0, f(0)=f'(0)=0, f'(\infty)=2$. The optimal co-ordinates may be found using (30); we choose $f_1(s)\equiv s$ and $f_2(s)=1$ in the optimal co-ordinate result ($8a, b$) and obtain the following specific co-ordinates:

$$\xi \equiv -\beta\gamma, \quad \eta \equiv 2\gamma\delta+2N\gamma^2\delta^2. \tag{32a, b}$$

In order to rewrite (31) in terms of ξ, η we require the following derivative for the correlation transformation $\partial\delta/\partial\eta$. We proceed by differentiating ($32b$) implicitly with respect to η to find

$$1 = 2\gamma_\eta\delta+2\gamma\delta_\eta+4N(\gamma^2\delta\delta_\eta+\delta^2\gamma\gamma_\eta). \tag{33}$$

We are interested in δ_η at $\eta=0$ (corresponding to $\delta=0$); hence (33) implies

$$(\delta_\eta)_{\eta=0} = \tfrac{1}{2}\gamma(\xi, 0). \tag{34}$$

The direct application of the correlation transformation (retaining implicit expressions) gives

$$\psi \sim \epsilon\left[\gamma(\xi,0)f\left(\frac{\bar{\eta}}{2\gamma(\xi,0)}\right)+\ldots\right]. \tag{35}$$

In order to verify our theory, we examine (35) for large $\bar{\eta}$; this requires the asymptotic expansion for f, which reads $f(s)\sim 2s-\beta+\ldots$. The introduction of this expansion gives

$$\psi \sim \eta+\epsilon[-\gamma(\xi,0)\beta]+\ldots \tag{36}$$

from (35), for large $\bar{\eta}$. The bracket term is identical to ξ or ψ_2 and $\eta \equiv \psi_1$. Consequently, we have verified that the solution in ξ, η, is indeed, an optimal one as the theory states. The co-ordinate system found (32) is 'parabolic' with respect to γ, δ, as well as parameter-dependent upon N.

Initial development of the boundary layer on a circular cylinder started from rest

Wang (1967) investigated the flow past a circular cylinder that is started impulsively from rest. This example is particularly interesting since it is an unsteady boundary-layer problem with complex solutions in *three* independent variables. In addition, the geometry is particularly nice with a clearly defined length – the radius of the circular cylinder. We avoid the obvious geometry sketch and proceed with the expansions for the stream function (incompressible flow). They read

$$\psi \sim \psi_1(r,\theta)+\epsilon\psi_2(r,\theta,t)+\ldots, \tag{37a}$$

$$\psi \sim \epsilon\Psi_1(\overline{r^*},\theta,t)+\ldots, \tag{37b}$$

where r is the dimensionless radius, θ is the angular variable, t is the time variable, $\overline{r^*}=(r-1)/\epsilon$, and ϵ is defined as the small time parameter. This parameter is inversely proportional to the Reynolds number (cf. Wang 1967, equation (1)). The individual elements of the problem are as follows

$$\psi_1 = (r-r^{-1})\sin\theta, \tag{38a}$$

$$\psi_2 = -ar^{-1}(\gamma t)^{\frac{1}{2}}\sin\theta, \tag{38b}$$

$$\Psi_1 = 4(\nu t)^{\frac{1}{2}}[\zeta-\zeta\,\mathrm{erfc}\,\zeta+\pi^{-\frac{1}{2}}(e^{-\zeta^2}-1)]\sin\theta, \tag{38c}$$

where $a = 4\pi^{-\frac{1}{2}}$, $\nu =$ constant, $\zeta = \frac{1}{2}\overline{r^*}(\nu t)^{-\frac{1}{2}}$. In § 4 we showed that two 'types' of optimal co-ordinate are possible for unsteady flow.

(*a*) We will first utilize (17) to choose an optimal set of co-ordinates (time-dependence passive). Hence, let

$$\xi = \psi_2(-a)^{-1}(\nu t)^{-\frac{1}{2}} = (r^*+1)^{-1}\sin\theta, \tag{39a}$$

$$\eta = -\frac{a\psi_1}{4\psi_2} = \frac{(1+r^*)^2-1}{4(\nu t)^{\frac{1}{2}}}, \quad \tau \equiv t, \tag{39b, c}$$

where $r^* \equiv r-1$.

In order to verify that (39) is an optimal co-ordinate set, we must determine an implicit derivative $-r^*_\eta$, i.e.

$$1 = \frac{2(1+r^*)r^*_\eta}{4(\nu t)^{\frac{1}{2}}} \quad \text{or} \quad r^*_\eta|_{\eta=0} = 2(\nu t)^{\frac{1}{2}}. \tag{40}$$

Also, from (39*a*)

$$\xi(r^* = 0, \theta, t) = \sin\theta. \tag{41}$$

From the correlation transformation r^* becomes $\overline{\eta}r^*_\eta|_{\eta=0} = \overline{\eta}2(\nu t)^{\frac{1}{2}}$. The previous similarity variable $\zeta \equiv \overline{\eta}$ and, hence the boundary-layer solution reads as follows

$$\psi \sim \epsilon\Psi_1 = \epsilon 4(\nu t)^{\frac{1}{2}}\,\xi[\overline{\eta} - \overline{\eta}\operatorname{erfc}\overline{\eta} + \pi^{-\frac{1}{2}}(e^{-\overline{\eta}^2}-1)]. \tag{42}$$

Specific notice should be taken of the tri-separated form of the solution in the $(\xi, \overline{\eta}, t)$-system. The expansion of the square bracket for large $\overline{\eta}$ gives

$$\psi \sim \epsilon 4(\nu t)^{\frac{1}{2}}\,\xi[\overline{\eta} - \pi^{-\frac{1}{2}}] + \ldots\,. \tag{43}$$

Substitution for (ξ, η) gives $\psi \sim \psi_1 + \epsilon\psi_2 + \ldots$, hence the co-ordinates (39) are optimal.

(*b*) We now utilize (18) to choose optimal co-ordinates (surface-co-ordinate dependence passive). Let

$$\xi = \theta, \quad \eta = \psi_1/4\sin\theta = \tfrac{1}{4}[(1+r^*)-(1+r^*)^{-1}], \tag{44a, b}$$

$$\tau = \psi_2/(-a\sin\theta) = (\nu t)^{\frac{1}{2}}/(r^*+1). \tag{44c}$$

From (44), $r^*_\eta|_{\eta=0} = 2$ and $\tau(r^* = 0, \theta, t) = (\nu t)^{\frac{1}{2}}$. Hence, $\overline{r^*}$ becomes $2\overline{\eta}$ in the correlation transformation and

$$\Psi_1 = 4\tau[\overline{\eta}/\tau - (\overline{\eta}/\tau)\operatorname{erfc}(\overline{\eta}/\tau) + \pi^{-\frac{1}{2}}(e^{-(\overline{\eta}/\tau)^2}-1)]\sin\theta.$$

The large-$\overline{\eta}$ expansion of this solution is, as before,

$$\Psi_1 \sim 4\tau(\overline{\eta}/\tau - \pi^{-\frac{1}{2}})\sin\theta + \ldots\,. \tag{45}$$

If we substitute for τ and $\overline{\eta}$, we find that once again $\psi \sim \psi_1 + \epsilon\psi_2 + \ldots$. The comparison between co-ordinates (39) and (44) is enlightening. It appears that with respect to optimal co-ordinates the time variable and the timelike variable (streamwise co-ordinate) play the same role as that alluded to in § 4. Furthermore, we have demonstrated that an extra variable (such as the time variable in the present context) does not lead to significant changes from Kaplun's work with two co-ordinates.

7. Concluding remarks

Kaplun (1954) limited his analysis of the role of co-ordinate systems in boundary-layer theory to incompressible, steady, two-dimensional (plane) flow past a solid body without separation, and with an irrotational free stream. Kaplun promised to remove those restrictions in a later paper (see Van Dyke 1975), but did not live to write it. In this paper we have attempted to point out the generality of Kaplun's results. The underlying feature that allows one to obtain optimal co-ordinates is the use of a stream function. This feature is continually utilized in this paper to remove many of the restrictions of Kaplun (1954). Two specific examples were used to illustrate the generalizations to a rotational outer flow and to unsteady boundary layers.

The work of the present paper, as well as that of Kaplun, focussed upon first-order boundary-layer theory. This restriction was removed by Legner (1971). The generalization of the results of the present paper to higher-order boundary-layer approximations will be the subject of a subsequent paper. Some discussion on this generalization appears in Van Dyke (1975).

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